

Examination projects in quantum field theory

Spring term

D. V. Naumov

<https://neutrinohit.github.io/en/qft-exams.html>

The examination in the course “Quantum Field Theory for Experimentalists” takes the form of mini-projects. Each student receives an individual problem and about a week to prepare. Each project is a small piece of research: theory, calculation, simulation and comparison with experiment.

Goals

- Applying the knowledge gained to real scientific problems.
- Mastering the tools of research: $\LaTeX$, git, python.
- Developing the skills of working together on projects.
- Critical thinking and creativity: the projects encourage unconventional solutions and well-argued conclusions.

Structure of the report

1. Introduction: motivation, a short review of the literature, statement of the problem.
2. Theory: derivations, calculations, formulas.
3. Practical part: simulation, data analysis, calculations.
4. Discussion: plots, tables, interpretation of the results.
5. Conclusions: the main results and how the work could be continued.
6. References: all sources used.

Technical aspects

- Write the project in $\LaTeX$; the online service overleaf.com is convenient.
- Feynman diagrams: the tikz-feynman package.
- Plots: the matplotlib library.

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1. Can masses depend on position?

In the Higgs mechanism the masses of the gauge and fermion fields are determined by the vacuum expectation value (VEV) of a scalar field. A natural question arises: *can this expectation value be made position dependent?* At first sight a configuration such as

$$|\phi(x)|^2 = \frac{v^2}{2}, \quad \phi(x) = \frac{v}{\sqrt{2}} \begin{pmatrix} \sin \theta(x) \\ \cos \theta(x) \end{pmatrix}$$

sits at the minimum of the potential and seems to be allowed. Then the masses of particles could vary from point to point. In this project you will find out why this is not so.

Theory

✓ Check that every configuration with $|\phi(x)|^2 = \frac{v^2}{2}$ gives zero potential. Show that for an arbitrary $\theta(x)$ the Lagrangian acquires an additional energy density from the derivatives:

$$\mathcal{E}_{\text{grad}} \sim (\partial_\mu \theta)^2.$$

✓ Compute the contribution of the configuration $\theta(x) = \theta_0 \sin(kx)$ to the total energy of the field. Show that the gradient term is non-negative and is minimal for $\partial_\mu \theta = 0$.

✓ Estimate the typical energy density of such configurations and compare it with the energy density of the Universe today ($\sim 10^{-47} \text{ GeV}^4$). What would it take for the effect to be noticeable?

✓ Estimate the characteristic time in which such a configuration relaxes to the vacuum (the relaxation time). What is it for $k^{-1} \sim 1 \text{ Mpc}$? Could remnants of this relaxation be observed today?

✓ What happens to the Goldstone bosons when the symmetry is local? Explain how $\theta(x)$ is “eaten” by the gauge field and becomes the longitudinal polarization of a massive gauge boson. Why, for a local symmetry, can $\theta(x)$ be removed by a gauge transformation, so that the masses of particles do not depend on it?

✓ Discuss how, if the masses of particles did depend on position, this would affect

- the spectra of distant atoms;
- the ratios of transition frequencies in different parts of the Universe;
- gravitational lensing and the observed composition of stars.

Observational constraints

✓ Study the results of

- searches for variations of fundamental constants: see the data from quasar spectra (for example, [1]);
- experiments with atomic clocks (frequency differences when moving through the gravitational potential);
- constraints on domain walls and other traces of an inhomogeneous phase.

✓ Draw a conclusion: is $|\phi(x)| = \text{const}$ really the only possible state of the Higgs field in the Universe? Or is there a chance that somewhere the electron mass was different?

2. The Standard Model at the Z pole

Experiments at the Large Electron–Positron collider (LEP) at CERN were decisive for precision studies of the electroweak interactions.

(i) Consider the decay $Z \rightarrow f\bar{f}$. Neglect the masses of the fermions.

a) Show that the total polarization of the fermion pair can be found by measuring the asymmetry

$$A_{LR}^f = \frac{\Gamma(Z \rightarrow f_L \bar{f}_R) - \Gamma(Z \rightarrow f_R \bar{f}_L)}{\Gamma(Z \rightarrow f_L \bar{f}_R) + \Gamma(Z \rightarrow f_R \bar{f}_L)}. \quad (1)$$

b) Express A_{LR}^f in terms of the electric charge Q_f , the third component of the weak isospin T_3^f of the fermion and $\sin^2 \theta_w$ in the Standard Model.

c) Compute A_{LR}^f for the charged leptons, the neutrinos and the quarks.

(ii) Define the forward–backward asymmetry of the process $e^+ e^- \rightarrow f\bar{f}$ as

$$A_{FB}^f = \frac{\int_0^1 d\cos\theta \frac{d\sigma}{d\cos\theta} - \int_{-1}^0 d\cos\theta \frac{d\sigma}{d\cos\theta}}{\int_0^1 d\cos\theta \frac{d\sigma}{d\cos\theta} + \int_{-1}^0 d\cos\theta \frac{d\sigma}{d\cos\theta}}. \quad (2)$$

d) Find A_{FB}^f as a function of $\sqrt{s}$. Plot it.

e) Show that at $\sqrt{s} = m_Z$ the asymmetries A_{LR}^f and A_{FB}^f are related by

$$A_{FB}^f = \frac{3}{4} A_{LR}^e A_{LR}^f. \quad (3)$$

f) Show that at $\sqrt{s} = m_Z$

$$\sigma(e^+ e^- \rightarrow f\bar{f}) = \frac{12\pi}{m_Z^2} \frac{\Gamma(Z \rightarrow e^+ e^-) \Gamma(Z \rightarrow f\bar{f})}{\Gamma^2(Z \rightarrow \text{all})}. \quad (4)$$

(iii) Discuss how $\sigma(e^+ e^- \rightarrow f\bar{f})$ and $\Gamma(Z \rightarrow \text{all})$ would change if there were a fourth type of neutrino.

3. Solar neutrinos

Nuclear reactions in the Sun produce electron neutrinos ν_e . Measurements of the solar neutrino flux proved that nuclear reactions take place in the Sun. The deficit of the measured flux $\Phi(E)_{\nu_e}$ was explained by Mikheyev, Smirnov and Wolfenstein as a result of neutrino oscillations in the matter of the Sun. This made it possible to measure the mixing angle θ_{12} and the difference of the squared neutrino masses $\Delta m_{21}^2 \equiv m_2^2 - m_1^2$.

- (i) Compute the cross sections of the reactions $\nu_e + e^- \rightarrow e^- + \nu_e$ (W and Z exchange) and $\nu_\mu + e^- \rightarrow \nu_\mu + e^-$ (Z exchange). Plot them as functions of the neutrino energy for $E_\nu \in (1, 10)$ MeV.
- (ii) In the Super-Kamiokande experiment (Japan), solar neutrino interactions are observed by detecting the Cherenkov light of the recoil electrons. In a model with two neutrino types, ν_e and ν_μ , neutrino oscillations redistribute the fluxes, $\Phi_{\nu_e} \leftrightarrow \Phi_{\nu_\mu}$. How did physicists conclude that there is a deficit of recoil electrons, if electrons appear in both reactions, $\nu_e + e^- \rightarrow e^- + \nu_e$ and $\nu_\mu + e^- \rightarrow \nu_\mu + e^-$? Support your answer with numerical estimates.
- (iii) Compute the expected number of recoil-electron events per year in a water detector of mass 50 kt, assuming a solar neutrino flux $\Phi = 5.7 \cdot 10^6 \text{ cm}^{-2} \text{ s}^{-1}$ and a mean energy $\langle E_\nu \rangle = 8.5$ MeV.
- (iv) The electron neutrino $|\nu_e\rangle = \cos \theta |\nu_1\rangle + \sin \theta |\nu_2\rangle$ and the muon neutrino $|\nu_\mu\rangle = -\sin \theta |\nu_1\rangle + \cos \theta |\nu_2\rangle$ are coherent superpositions of the mass states $|\nu_{1,2}\rangle$. Because of the Mikheyev–Smirnov–Wolfenstein effect, for energies $E_\nu \geq 5$ MeV what leaves the Sun is neither $|\nu_e\rangle$ nor $|\nu_\mu\rangle$ but the pure mass state $|\nu_2\rangle$. Revisit items (i), (ii), (iii) in the light of this fact and discuss your conclusions.

4. Relic neutrinos

According to the modern cosmological model of the birth and evolution of the Universe, we are surrounded by the so-called *relic neutrinos*. Their temperature today is $T_\nu \approx 1.95$ K ($kT_\nu \approx 1.7 \cdot 10^{-4}$ eV), which makes them very hard to observe. Nevertheless, let us think of something. A Nobel Prize is guaranteed!

(i) Compute the cross section of the reaction $\nu_e + n \rightarrow p + e^-$ at the energy of a relic neutrino. Compute the mass of free neutrons needed to detect 100 such events per year. Do we have a suitable source of free neutrons of the required mass?

If not, suggest an atomic nucleus that could be used instead of free neutrons.

(ii) Consider another project: a particle accelerator. If an electron or a muon is accelerated in it to a high enough energy, the lepton can disappear after interacting with a relic antineutrino, producing a pion: $\bar{\nu}_\ell + \ell^- \rightarrow \pi^-$.

a) Compute the lepton energy needed for this reaction to occur.

b) Compute the corresponding cross section.

c) Estimate the luminosity of the accelerator needed to detect a statistically significant number of pions.

(iii) Is there a hidden Glashow resonance in the reaction $\bar{\nu}_\ell + \ell^- \rightarrow \pi^-$? (Discuss this idea with the colleague working on that project.) What could follow from it?

(iv) Propose your own idea for detecting relic neutrinos and support it with calculations.

5. Reactor antineutrinos

Nuclear reactors of power plants are intense sources of antineutrinos. A reactor emits isotropically about $2 \cdot 10^{20}$ antineutrinos per second for every GW of thermal power. Consider a project of experiments with reactor neutrinos.

- (i) Compute the cross section of the interaction $\bar{\nu}_e + p \rightarrow e^+ + n$.
- (ii) Estimate the number of such interactions per year for $E_{\bar{\nu}} = 4$ MeV from a typical reactor of 3 GW. The target is water of mass 20 t. The distance between the detector and the reactor is 2 km.
- (iii) Estimate the detector mass needed to detect 10^5 events $\bar{\nu}_e + p \rightarrow e^+ + n$ at a distance $L = 50$ km from reactors with a total power of 30 GW. Take neutrino oscillations into account.

6. The neutral pion

In e^+e^- collisions one can observe collisions of almost real photons emitted by the electrons and positrons at small angles to the beam axis. In such collisions the photons can produce a neutral pion. The cross section of this reaction is proportional to the decay width $\Gamma(\pi^0 \rightarrow \gamma\gamma)$ and provides a way to measure it.

(i) Compute the cross section of the reaction $\gamma + \gamma \rightarrow \pi^0$ in QED:

$$\sigma(\gamma + \gamma \rightarrow \pi^0) = \frac{8\pi^2 \Gamma(\pi^0 \rightarrow \gamma\gamma)}{m_\pi} \delta((p + k)^2 - m_\pi^2), \quad (5)$$

where p, k are the 4-momenta of the photons (Low's formula; the photons are unpolarized).

(ii) Check that the matrix element of the reaction $\gamma + \gamma \rightarrow \pi^0$ is nonzero only when the helicities of the photons are equal (for linear polarizations, when they are mutually perpendicular).

(iii) Check that in the decay $\pi^0 \rightarrow \gamma\gamma$ the combination $(\mathbf{E}_1 + \mathbf{E}_2) \cdot (\mathbf{B}_1 + \mathbf{B}_2)$ is nonzero only for the same photon polarizations.

7. The Glashow resonance

The neutrino telescopes IceCube, Baikal-GVD, KM3NeT and P-ONE aim to identify astrophysical sources of ultra-high-energy neutrinos.

- (i) Without neglecting q^2 compared with m_W^2 in the W -boson propagator, compute the cross sections of the deep inelastic reactions $\nu_\ell + N \rightarrow \ell^- + X$ and $\bar{\nu}_\ell + N \rightarrow \ell^+ + X$ (X stands for hadrons). Plot them as functions of the neutrino energy. Use, for example, the 1hpdf library for the quark distributions.
- (ii) Estimate the absorption length of neutrinos and antineutrinos. Find the energy above which the Earth is no longer transparent to them.
- (iii) Compute the cross section of the reaction $\bar{\nu}_e + e^- \rightarrow W^-$ (the Glashow resonance). Why is it called so? Compare this cross section with the cross sections from (i). Is there a similar resonance for ν_e ? Why?
- (iv) Compute the cross sections of the reactions $\bar{\nu}_e + e^- \rightarrow \bar{\nu}_\ell + \ell^-$ and $\bar{\nu}_e + e^- \rightarrow X$ in the region of the Glashow resonance. Compare them with the cross sections from (i) on one plot.
- (v) What can the observation of the Glashow resonance tell us? Give a short review of the experimental status.
- (vi) Estimate the volume of a water telescope that would detect one such event per year.

8. Neutral kaons

The system of neutral kaons is a unique laboratory for studying quantum oscillations and CP violation. In this project we concentrate on a precise measurement of the parameter ε , which characterizes the violation of CP invariance.

Theory

✓ What are K^0 , $\bar{K}^0$, K_S , K_L , K_1 , K_2 ?

Relate these states to one another, for example:

$$\begin{aligned} |K_L\rangle &= \frac{1}{\sqrt{1+|\varepsilon|^2}}(|K_2\rangle + \varepsilon|K_1\rangle), \\ |K_S\rangle &= \frac{1}{\sqrt{1+|\varepsilon|^2}}(|K_1\rangle + \varepsilon|K_2\rangle). \end{aligned} \quad (6)$$

✓ Derive the transition probabilities $P_{K^0 \rightarrow K^0}(t)$ and $P_{K^0 \rightarrow \bar{K}^0}(t)$ in two cases: CP symmetry conserved and CP symmetry violated.

✓ Compute the mass difference of the neutral kaons Δm in the SM. Explain the essence of the GIM mechanism. Estimate the mass of the c quark.

✓ Assuming CP symmetry is conserved, obtain the expressions for the decay rates

$$\begin{aligned} R_+ &= \Gamma(K^0 \rightarrow \pi^- e^+ \nu_e), \\ R_- &= \Gamma(K^0 \rightarrow \pi^+ e^- \bar{\nu}_e), \\ \bar{R}_+ &= \Gamma(\bar{K}^0 \rightarrow \pi^+ e^- \bar{\nu}_e), \\ \bar{R}_- &= \Gamma(\bar{K}^0 \rightarrow \pi^- e^+ \nu_e). \end{aligned}$$

Compute the decay asymmetry

$$A = \frac{(R_+ + \bar{R}_+) - (R_- + \bar{R}_-)}{(R_+ + \bar{R}_+) + (R_- + \bar{R}_-)} = \frac{2e^{-(\Gamma_S + \Gamma_L)t/2} \cos \Delta m t}{e^{-\Gamma_S t} + e^{-\Gamma_L t}}.$$

An example of pseudo-data for the decay asymmetry is available at [2]. It was generated with $\tau_S = 8.954 \cdot 10^{-11}$ s, $\tau_L = 5.116 \cdot 10^{-8}$ s and a value of Δm that you have to reconstruct.

From now on we assume that CP symmetry is violated.

A beam of neutral kaons studied at $t \gg \tau_S$ consists mainly of K_L .

✓ Show that

$$\begin{aligned} \Gamma(K_L \rightarrow \pi^+ e^- \bar{\nu}_e) &\propto |\langle \bar{K}^0 | K_L \rangle|^2 \propto |1 - \varepsilon|^2, \\ \Gamma(K_L \rightarrow \pi^- e^+ \nu_e) &\propto |\langle K^0 | K_L \rangle|^2 \propto |1 + \varepsilon|^2, \end{aligned}$$

and compute the asymmetry

$$A = \frac{\Gamma(K_L \rightarrow \pi^- e^+ \nu_e) - \Gamma(K_L \rightarrow \pi^+ e^- \bar{\nu}_e)}{\Gamma(K_L \rightarrow \pi^- e^+ \nu_e) + \Gamma(K_L \rightarrow \pi^+ e^- \bar{\nu}_e)}$$

using the tabulated values.

✓ Show how this asymmetry can be used to explain the sign of the charge of the electrons in our atoms.

✓ Using box diagrams, show that the inequality $\Gamma(K_L \rightarrow \pi^- e^+ \nu_e) \neq \Gamma(K_L \rightarrow \pi^+ e^- \bar{\nu}_e)$ requires the CKM quark mixing matrix to be complex.

✓ Prove that the ratio of the decay rates is

$$\frac{\Gamma(K_L \rightarrow \pi^+ \pi^-)}{\Gamma(K_S \rightarrow \pi^+ \pi^-)} \approx |\varepsilon|^2. \quad (7)$$

Experimental part: pseudo-data

The table gives the data of an experiment. Reconstruct $|\varepsilon|$, taking into account

- the detection efficiency of $\pi^+\pi^-$ in the decays of K_S and K_L ;
- the contribution of background events.

Table 1: Experimental data

Decay	Number of events	Efficiency (%)	Background (%)
$K_S \rightarrow \pi^+\pi^-$	12 000	95	1
$K_L \rightarrow \pi^+\pi^-$	72	90	10

Systematic uncertainties

Assuming systematic uncertainties

- of 5% in the number of background events,
- of 2% in the detection efficiency,

estimate $|\varepsilon|$ and give its statistical and systematic uncertainties.

9. Decay of the Higgs boson

Consider the decay of the Higgs boson into a fermion pair $f\bar{f}$.

The quantum state of the fermions has the form

$$|f\bar{f}\rangle = \int \frac{d\mathbf{p}_1}{(2\pi)^3 \sqrt{2E_1}} \frac{d\mathbf{p}_2}{(2\pi)^3 \sqrt{2E_2}} \tilde{\psi}(\mathbf{p}_1, s_1; \mathbf{p}_2, s_2) |f(\mathbf{p}_1, s_1), \bar{f}(\mathbf{p}_2, s_2)\rangle, \quad (8)$$

where $\tilde{\psi}$ is the two-particle wave function of the fermions in the momentum representation and s_i are the spin 4-vectors. In the rest frame of a fermion $s = (0, \boldsymbol{\xi})$.

✓ Find $\tilde{\psi}$ using the matrix element

$$\langle f(\mathbf{p}_1, s_1), \bar{f}(\mathbf{p}_2, s_2) | S - 1 | H \rangle.$$

✓ Now compute the wave function in the coordinate representation:

$$\psi(\mathbf{x}_1, \mathbf{x}_2) = \int \frac{d\mathbf{p}_1}{(2\pi)^3} \frac{d\mathbf{p}_2}{(2\pi)^3} \tilde{\psi}(\mathbf{p}_1, s_1; \mathbf{p}_2, s_2) e^{+i(\mathbf{p}_1 \cdot \mathbf{x}_1 + \mathbf{p}_2 \cdot \mathbf{x}_2)}.$$

Show that $\psi(\mathbf{x}_1, \mathbf{x}_2)$ depends on the difference of the coordinates $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$ and compute the wave function $\psi(\mathbf{r})$.

✓ Expand the wave function in spherical harmonics and study how it depends on $\boldsymbol{\xi}_1 \cdot \hat{\mathbf{r}}$, $\boldsymbol{\xi}_2 \cdot \hat{\mathbf{r}}$, $\boldsymbol{\xi}_1 \cdot \boldsymbol{\xi}_2$. Interpret the result.

10. Neutrino radiation hazard

Every second an enormous number of neutrinos passes through the Earth and its inhabitants. Estimate the effect of these fluxes on a human being in terms of radiation dose.

- ✓ Compute the cross section of neutrino scattering on an electron and the deep inelastic cross section on a nucleon (use the parton distributions of LHAPDF).
- ✓ Use the fluxes of solar, atmospheric and relic neutrinos from the literature.
- ✓ Compute the number of interactions per year in a human body of mass 70 kg and the energy deposited.
- ✓ Convert this number of interactions into an equivalent dose (Sv/year) using the approximation

$$D \approx N_{\text{events}} \cdot \langle E_{\text{dep}} \rangle \cdot \frac{Q}{m},$$

where $Q = 1$ is the weighting factor for secondary electrons and muons, $m = 70$ kg and $\langle E_{\text{dep}} \rangle$ is the mean energy deposited in the body in one interaction.

- ✓ Compare this dose with the natural background and the permitted limits. Draw a conclusion: are neutrinos dangerous?

11. When the vacuum becomes a prism

Renormalization is sometimes seen as a formal trick. Yet it uncovers a new layer of reality with real physical effects. One of them is the **birefringence of the vacuum** in a strong electromagnetic field, which follows from the Lagrangian of Euler and Heisenberg [3]. In this project you will see that quantum fluctuations make the vacuum an optically anisotropic medium.

✓ In classical electrodynamics the Lagrangian of the field is

$$L_0 = \frac{1}{8\pi}(\mathbf{E}^2 - \mathbf{B}^2).$$

Quantum corrections due to virtual electron–positron pairs lead to nonlinear effects. At low energies the effective Lagrangian (the Euler–Heisenberg Lagrangian) takes the form

$$L_{\text{eff}} = L_0 + \frac{e^4}{360\pi^2 m_e^4} \left[(\mathbf{E}^2 - \mathbf{B}^2)^2 + 7(\mathbf{E} \cdot \mathbf{B})^2 \right].$$

(a) Derive the expressions for the displacement and the magnetizing field:

$$\mathbf{D} = 4\pi \frac{\partial L_{\text{eff}}}{\partial \mathbf{E}}, \quad \mathbf{H} = -4\pi \frac{\partial L_{\text{eff}}}{\partial \mathbf{B}}.$$

(b) Compare the constitutive relations you obtain with those of the vacuum. What has changed?

✓ Linearize the equations, assuming that the field consists of a background $\mathbf{E}_0, \mathbf{B}_0$ and a small perturbation $\delta\mathbf{E}, \delta\mathbf{B}$. Obtain the permittivity and permeability tensors of the medium.

✓ Write Maxwell's equations for the wave perturbation. Find the dispersion relation for a wave propagating along the x axis, assuming that the constant magnetic field is directed along the z axis:

$$\mathbf{B}_0 = B_0 \mathbf{e}_z.$$

✓ Consider two cases:

(a) the electric field of the wave is parallel to $\mathbf{B}_0$ (polarization along the z axis);

(b) the electric field of the wave is perpendicular to $\mathbf{B}_0$ (polarization along the y axis).

✓ Find the refractive indices $n_{\parallel}$ and $n_{\perp}$ and their difference

$$\Delta n = n_{\parallel} - n_{\perp}.$$

✓ Propose an experiment to observe vacuum birefringence. Which effect could be used: interference, rotation of the polarization, a phase shift? What fields and what sensitivity are needed?

12. The short life of orthopositronium

Positronium is a bound state of an electron and a positron, similar to a hydrogen-like atom. Its states are classified by the total spin:

- *parapositronium* (spin 0) decays mainly into two photons;
- *orthopositronium* (spin 1) decays into three.

In this project you will study the decay mechanism of orthopositronium, estimate its lifetime and see how QED describes the decay of composite particles.

✓ Consider the decay of orthopositronium into three photons, $o\text{-Ps} \rightarrow 3\gamma$, assuming that the electron-positron system is at rest (the approximation $v \ll c$).

- Draw all six tree-level diagrams of the process. Explain why there are exactly six. Write the full matrix element $\mathcal{M}$, taking into account that the photons are identical.
- Compute the squared matrix element $|\mathcal{M}|^2$, averaged over the initial and summed over the final spins, in the approximation of fermions at rest ($\mathbf{p} = \mathbf{p}' = 0$).
- Find the differential decay rate in the energy of one of the photons,

$$\frac{d\Gamma}{d\omega_1},$$

where ω_1 is the energy of the first photon in the center-of-mass frame. Take the symmetry of the process into account.

- Integrate and obtain the total decay width $\Gamma(o\text{-Ps} \rightarrow 3\gamma)$. Using

$$\tau = \frac{\hbar}{\Gamma},$$

find the lifetime of orthopositronium in vacuum.

- Modify the calculation assuming that the electron and the positron are in the ground state of a hydrogen-like atom (wave function $\psi(0) \neq 0$). How does this change the result?

13. Vavilov–Cherenkov radiation

An electron moves in a homogeneous, transparent, nonmagnetic medium with refractive index $n > 1$. If the speed of the electron is $v > 1/n$, the decay $e(p) \rightarrow e(p') + \gamma(k)$ becomes possible, which gives rise to Vavilov–Cherenkov radiation. The photon energy is $\omega_k = |\mathbf{k}|/n$. Its polarization is $\epsilon_\lambda^\mu(\mathbf{k}) = (0, \epsilon_\lambda(\mathbf{k}))$.

✓ Show that the squared matrix element, summed over the spins of the final electron and averaged over the spins of the initial electron, is

$$|\mathcal{M}|^2 = 4\pi\alpha \left(4 (\mathbf{p} \cdot \epsilon_\lambda(\mathbf{k}))^2 - k^2 \right).$$

This expression shows that the photon is polarized in the scattering plane spanned by the vectors $\mathbf{p}$ and $\mathbf{k}$.

✓ Show that the spectral and angular distribution of the radiation is

$$\frac{d^2\Gamma}{d\omega d\Omega_\gamma} = \frac{|\mathcal{M}|^2}{32\pi^2 v E^2} \delta \left(\cos \theta - \frac{1}{vn} - \frac{\omega(n^2 - 1)}{2vnE} \right),$$

which shows that the polar emission angle of the photon (the z axis is along the momentum $\mathbf{p}$ of the initial electron) is fixed by the energies of the initial electron and the final photon:

$$\cos \theta \equiv \cos \theta_0 = \frac{1}{vn} + \frac{\omega(n^2 - 1)}{2vnE}.$$

✓ Show that the spectral distribution of the radiation is

$$\frac{d\Gamma}{d\omega} = \alpha \left[v \sin^2 \theta_0 + \frac{\omega^2(n^2 - 1)}{2vE^2} \right].$$

The future Nobel laureate Vitaly Lazarevich Ginzburg solved this problem in 1940.

Neglecting terms of order ω/E , we recover the classical results of two other Nobel laureates, I. E. Tamm and I. M. Frank:

$$\frac{d\Gamma}{d\omega} = \alpha v \sin^2 \theta_0, \quad \cos \theta_0 = \frac{1}{vn}.$$

14. The cosmological constant

- ✓ Show that a nonzero (infinite!) vacuum energy density of a quantum field redefines the cosmological constant in Einstein's equations.
- ✓ Discuss how the solution describing the evolution of a homogeneous and isotropic Universe changes depending on the contributions of fermion and boson fields to the vacuum energy.
- ✓ Study the literature on modern experimental measurements of the cosmological constant in the model of a homogeneous and isotropic Universe. State what properties a system must have to explain the observed accelerated expansion of the Universe.
- ✓ Consider the simplest supersymmetric model, the Wess–Zumino model [4], which contains one complex scalar ϕ and one Majorana fermion ψ . Obtain the mean vacuum energy density. Does supersymmetry solve the cosmological constant problem?

15. The functional integral

- ✓ Consider the functional integral for a scalar field. Formulate the rules for obtaining Feynman diagrams from the generating functional.
- ✓ Obtain the expressions for the vacuum energy (the energy density of the zero-point oscillations) and draw the corresponding Feynman diagrams for the $\lambda\phi^4$ model.
- ✓ Show that vacuum loops (“bubbles”) do not contribute to scattering amplitudes.
- ✓ Compute the vacuum energy density for scalar, fermion, and massive and massless vector fields. Write the final expression for the mean vacuum energy density including quantum corrections.

16. Grand Unified Theories

This project is devoted to attempts to unify the fundamental interactions (strong, weak and electromagnetic) within simple gauge groups such as $SU(5)$. The tasks include building the model, studying the mechanism of spontaneous symmetry breaking, predicting proton decay and generating neutrino masses through the minimal dimension-5 operator.

✓ Consider the gauge group of the Standard Model (SM), $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$. Describe how the SM fields transform under the gauge transformations of this group. Construct the covariant derivatives of the SM fields. Carry out the spontaneous symmetry breaking

$$SU(3)_c \otimes SU(2)_L \otimes U(1)_Y \rightarrow SU(3)_c \otimes U(1)_{\text{em}}.$$

✓ Consider $SU(5)$ as a GUT group. Show the embedding $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y \subset SU(5)$. List the representations of $SU(5)$ and place the left-handed fermions in the $1 + \bar{5} + 10$ representations. The gauge bosons of $SU(5)$.

✓ Spontaneous symmetry breaking in $SU(5)$. The chain

$$SU(5) \rightarrow SU(3)_c \otimes SU(2)_L \otimes U(1)_Y \rightarrow SU(3)_c \otimes U(1)_{\text{em}}.$$

The fine-tuning problem of the Higgs parameters.

✓ Proton decay. Consider the process $p \rightarrow \bar{\nu}\pi^+$ in the effective low-energy Lagrangian and compute the mean lifetime of the proton. Discuss the role of the $B - L$ symmetry. Give the current experimental limits on the proton lifetime and describe how the detectors built to search for it work.

✓ Consider the seesaw mechanism for generating small neutrino masses. Restrict yourself to one generation of leptons. Construct the most general Yukawa part of the Lagrangian, including the Dirac and Majorana mass terms of the neutrino. Show that the low-energy effective action contains a single dimension-5 mass operator (the Weinberg operator). Obtain the prediction for the neutrino mass with a GUT scale of $\sim 10^{15}$ GeV.

17. Widths of unstable particles

In nonrelativistic quantum mechanics the instability of a particle is described by a complex potential or by a function $\psi(t) \sim e^{-iEt - \Gamma t/2}$. In quantum field theory the decay width arises from the imaginary part of the self-energy.

In this project you will trace how the width arises from diagrams and how it is related to observable quantities.

Theory

- ✓ Let the scalar field ϕ interact with a light field χ through

$$\mathcal{L}_{\text{int}} = -g\phi\chi^2.$$

Draw the second-order (in g) self-energy diagram of ϕ . Compute the imaginary part of the loop with the help of the optical theorem.

- ✓ Show that the field ϕ has the modified propagator

$$\frac{i}{p^2 - m^2 + im\Gamma}.$$

Explain the physical meaning of Γ by comparing with the exponential decay law.

- ✓ Show that the observed cross section near the resonance has the Breit–Wigner form

$$\sigma(E) \propto \frac{1}{(E - m)^2 + (\Gamma/2)^2}.$$

Derive this expression from the full amplitude with the field ϕ in the intermediate state.

- ✓ Explain why one cannot speak of a fixed mass of an unstable particle. What is the meaning of the diagram with the decay $A \rightarrow B\phi \rightarrow B\chi\chi$?
- ✓ Discuss how the decay width is related to mass measurements: why is the mass of the Z boson determined from the position of the peak of the cross section $e^+e^- \rightarrow f\bar{f}$, and the width from its shape?

A numerical example

- ✓ Compute the decay width of a hypothetical scalar field, $\phi \rightarrow \chi\chi$, for $m_\phi = 1$ GeV, $m_\chi = 0.2$ GeV, $g = 1$ GeV.
- ✓ Plot $\sigma(E)$ near the resonance and show how the shape of the peak changes as Γ grows. How does the width depend on the mass and the coupling?
- ✓ Find the widths of the real resonances ρ , K^* , Δ in the PDG tables. Compare with the Breit–Wigner formula. Why does it work poorly in some cases?

18. Polarizing glasses and QED

Polarizing glasses transmit light of only one linear polarization. They are useful because scattered and reflected light is partially polarized: when light scatters on electrons, the intensity of the scattered radiation depends on its polarization and on the scattering angle. In this project you will show that quantum electrodynamics predicts this dependence exactly.

✓ Consider the scattering of a photon on a free electron. Find the differential cross section $d\sigma/d\Omega$ as a function of the polarizations of the incident and scattered photons.

✓ Assuming the incident light is unpolarized, show that the scattered photon is partially linearly polarized. Obtain the degree of polarization $\mathcal{P}(\theta)$ as a function of the angle between the incident and the scattered photon:

$$\mathcal{P}(\theta) = \frac{\sin^2 \theta}{1 + \cos^2 \theta}.$$

Hint. The problem becomes much simpler if the energy of the electron hardly changes in the scattering.

✓ Plot $\mathcal{P}(\theta)$. At what scattering angle is the polarization maximal?

Applications in physics and astrophysics

- **Polarization of the sky.** Explain why sunlight scattered by the atmosphere becomes partially linearly polarized. At what scattering angle is the polarization maximal? Why does this matter for astronomical photography, navigation and polarizing glasses?
 - **Why do truck drivers wear polarizing glasses?** Explain why light reflected from wet asphalt is partially polarized. At what angle of incidence is the polarization maximal? Is a QED calculation (scattering on an electron) enough to describe this, or must something else be taken into account?
 - **The cosmic microwave background and the BICEP/Keck experiments.** Explain how the linear polarization of the cosmic microwave background arises from Compton scattering of photons on electrons in the early Universe.
 - **Polarization of starlight by interstellar dust.** Starlight that has passed through interstellar dust is partially polarized: elongated dust grains aligned by the magnetic field absorb one polarization more strongly than the other. How does this help to study the structure of magnetic fields in the Galaxy? How does this mechanism differ from polarization by scattering?
 - **Plasma diagnostics.** Thomson scattering is used to measure the temperature and density of electrons in laboratory plasmas. Explain how, in practice, the dependence of the degree of polarization $\mathcal{P}(\theta)$ can be used for diagnostics if the direction of the incident laser beam is known.
- ✓ Choose one of the applications and discuss it in detail.

References

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