

Examination projects in quantum field theory

Autumn term

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<https://neutrinohit.github.io/en/qft-exams.html>

The examination in the course “Quantum Field Theory for Experimentalists” takes the form of mini-projects. Each student receives an individual problem and about a week to prepare. Each project is a small piece of research: theory, calculation, simulation and comparison with experiment.

Goals

- Applying the knowledge gained to real scientific problems.
- Mastering the tools of research: $\LaTeX$, git, python.
- Developing the skills of working together on projects.
- Critical thinking and creativity: the projects encourage unconventional solutions and well-argued conclusions.

Structure of the report

1. Introduction: motivation, a short review of the literature, statement of the problem.
2. Theory: derivations, calculations, formulas.
3. Practical part: simulation, data analysis, calculations.
4. Discussion: plots, tables, interpretation of the results.
5. Conclusions: the main results and how the work could be continued.
6. References: all sources used.

Technical aspects

- Write the project in $\LaTeX$; the online service overleaf.com is convenient.
- Feynman diagrams: the tikz-feynman package.
- Plots: the matplotlib library.

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1. The quantum harmonic oscillator

Goal: Study the energy spectrum and the dynamics of a one-dimensional chain of quantum harmonic oscillators, including an analysis of its symmetries and a visualization of its evolution.

1. **The Hamiltonian:** Consider a one-dimensional chain of N identical atoms with spacing a between neighboring atoms. The Hamiltonian of the system is

$$\hat{H} = \sum_{i=1}^N \frac{\hat{P}_i^2}{2m} + \frac{k}{2} \sum_{i=1}^N (\hat{X}_{i+1} - \hat{X}_i)^2,$$

where

- m is the mass of an atom,
- $k = m\omega^2$ is the stiffness of the bond between neighboring atoms,
- $\hat{P}_i$ and $\hat{X}_i$ are the momentum and coordinate (displacement from equilibrium) operators of the i th atom.

Explain why only interactions between neighboring atoms are taken into account.

2. **Normal-mode operators:** Define the normal-mode operators

$$\begin{aligned}\hat{Q}_k &= \frac{1}{\sqrt{N}} \sum_{n=1}^N e^{ikan} \hat{X}_n, \\ \hat{\Pi}_k &= \frac{1}{\sqrt{N}} \sum_{n=1}^N e^{-ikan} \hat{P}_n,\end{aligned}$$

where $k = \frac{2\pi\ell}{Na}$, $\ell = 0, 1, \dots, N-1$.

- (a) Compute the commutators $[\hat{Q}_k, \hat{\Pi}_{k'}]$, $[\hat{Q}_k, \hat{Q}_{k'}]$, and $[\hat{\Pi}_k, \hat{\Pi}_{k'}]$.
- (b) Write the Hamiltonian $\hat{H}$ in terms of $\hat{Q}_k$ and $\hat{\Pi}_k$.

3. **Periodic boundary conditions:** Assume that the chain obeys periodic boundary conditions:

$$\hat{X}_{N+1} = \hat{X}_1.$$

- (a) Find the eigenfrequencies ω_k of the system, where k is the wave vector.
- (b) Show that the system is equivalent to a set of independent harmonic oscillators, one for each mode k .

4. **Dynamics of the atoms:**

- (a) Compute the wave function $\psi(t)$ of the system.
- (b) Make an animation of the mean displacements of the atoms $\langle X_n \rangle(t)$ for different initial conditions:
 - one atom displaced: $X_i(0) \neq 0$, all other $X_n(0) = 0$;
 - two atoms displaced: $X_i(0) \neq 0$, $X_k(0) \neq 0$, all other $X_n(0) = 0$.

5. **Symmetries and conserved charges:**

- (a) Find all symmetries of the problem (for example, translation symmetry and invariance under time shifts).
- (b) Using Noether's theorem, compute the conserved charges associated with each symmetry.
- (c) Explain the physical meaning of these charges (for example, energy, momentum, phase correlations of the modes).

2. The hydrogen atom

1. Compute the wave functions of the hydrogen atom for $n = 1$, $j = 1/2$. Plot them. Compare them with the solution of the Schrödinger equation for $n = 1$.
2. Compute and plot the emission spectrum of the hydrogen atom in Dirac's theory.
3. The energy levels of a hydrogen atom change in the gravitational field of a star. For the photons emitted by the atoms this leads to the gravitational redshift

$$z = \frac{\Delta\nu}{\nu} = \frac{GM}{Rc^2},$$

where G is the gravitational constant and M and R are the mass and the radius of the star. Compute $\Delta\nu/\nu$ for different types of stars. Which of them allow one to observe the *gravitational* redshift? Compare with experiment.

3. The $\pi^+\pi^-$ atom

Goal: Study the properties of the $\pi^+\pi^-$ atom taking into account the Coulomb and the strong interactions. Compute the binding energy and its correction due to the strong interaction, and compare them with experimental data.

1. **Coulomb binding energy:** Compute the energy of the 1s state with only the electromagnetic interaction taken into account. Estimate the Bohr radius.
2. **Strong-interaction correction:** Compute the correction ΔE to the binding energy of the 1s state from

$$\Delta E = \frac{\int_0^\infty |R_{1s}|^2 V_{\text{strong}}(r) r^2 dr}{\int_0^\infty |R_{1s}|^2 r^2 dr},$$

using three different potentials for the strong interaction:

- (a) *A contact potential (delta function):*

$$V_{\text{strong}}(r) = -\frac{4\pi a_0}{\mu} \delta^3(r),$$

where $\mu = m_\pi$ and $a_0 \approx 0.22 m_\pi^{-1} \approx 0.31 \text{ fm}$ is the $\pi\pi$ scattering length in the isospin $I = 0$ state.

- (b) *A finite-range potential (square well):*

$$V_{\text{strong}}(r) = \begin{cases} -V_0, & r \leq R_{\text{strong}}, \\ 0, & r > R_{\text{strong}}, \end{cases}$$

where $V_0 \sim 100 \text{ MeV}$, $R_{\text{strong}} \sim 1 \text{ fm}$.

- (c) *A Yukawa potential:*

$$V_{\text{strong}}(r) = -g^2 \frac{e^{-\mu_s r}}{r},$$

where g^2 is the strong coupling constant and $\mu_s \sim 1 \text{ fm}^{-1}$.

3. Compare the values of ΔE you obtain with experimental data.
4. Plot ΔE as a function of the parameters a_0 , V_0 , g^2 for the three potential models and compare with experiment.
5. Analyze the contribution of the strong interaction to the total binding energy.

4. Resonances and quasi-bound states

Goal: Study quasi-bound states and resonances in quantum mechanics, the relation between the width of a resonance and the lifetime, and the interpretation of resonances as poles of the S -matrix in the complex energy plane.

1. Consider the one-dimensional potential

$$V(x) = \begin{cases} -V_0, & |x| < a, \\ 0, & |x| > a. \end{cases}$$

2. Find the reflection and transmission coefficients $R(E)$, $T(E)$.
3. Show that there are resonance peaks with $T(E) \approx 1$ at $E > 0$.
4. Continue the energy into the complex plane, $E \rightarrow E - i\Gamma/2$, and find the conditions for the poles of the S -matrix.
5. Relate the width of the resonance Γ to the lifetime of the state,

$$\tau \sim \frac{1}{\Gamma}.$$

6. Plot numerically:

- $T(E)$;
- the trajectories of the poles as V_0 and a vary.

7. Discuss why resonances are more naturally interpreted in the language of scattering than in the language of “bound states”.

5. The Klein paradox

Goal: Study the scattering of a relativistic fermion on a potential step within the Dirac equation, understand the origin of the so-called Klein paradox, and show that it disappears when the current density and the spectrum of states are interpreted correctly.

We consider the one-dimensional scattering of a fermion of mass m on the potential step

$$V(x) = V_0 \theta(x)$$

within the single-particle Dirac equation, *without* second quantization.

1. The Dirac equation in 1+1 dimensions

Consider the stationary Dirac equation

$$\left[-i\gamma^0\gamma^1\partial_x + \gamma^0m + V(x) \right] \psi(x) = E\psi(x).$$

- Choose an explicit representation of the γ matrices in 1+1 dimensions.
- Write the equation in matrix form and find the plane-wave solutions in the regions $x < 0$ and $x > 0$.

2. Spectrum and wave vectors

For the regions

$$\text{I: } x < 0, \quad V = 0, \quad \text{II: } x > 0, \quad V = V_0,$$

find the corresponding wave vectors k_1, k_2 .

- Show that for

$$E < V_0 - m$$

the wave vector k_2 remains real.

- Explain why this regime does not exist in nonrelativistic theory.

3. Boundary conditions

Write the full solution as

$$\psi_I = \psi_{\text{in}} + R\psi_{\text{ref}}, \quad \psi_{II} = T\psi_{\text{tr}},$$

and impose continuity of the wave function at $x = 0$.

- Find the reflection coefficient R and the transmission coefficient T .
- Check that $R + T \neq 1$ with the naive definition of probability.

4. Current density

Define the Dirac current density

$$j^\mu = \bar{\psi}\gamma^\mu\psi, \quad j^x = \psi^\dagger\alpha\psi.$$

- Compute the current for the incident, reflected and transmitted waves.
- Show that current conservation,

$$j_{\text{in}} = j_{\text{ref}} + j_{\text{tr}},$$

holds *exactly*.

- (c) Explain why $|R| > 1$ is nevertheless possible.

5. Group velocity and interpretation

- (a) Compute the group velocity

$$v_g = \frac{dE}{dk}$$

in region II.

- (b) Show that the sign of the group velocity coincides with the sign of the current.
(c) Explain why the solution in the region $x > 0$ corresponds to states with negative energy relative to the local vacuum.

6. Physical interpretation

- (a) Explain how the paradox disappears when the solutions are interpreted as antiparticle states.
(b) Discuss why the language of particles and antiparticles appears already at the level of the single-particle Dirac equation.
(c) Compare with the nonrelativistic case (the Schrödinger equation).

7. Numerical analysis

Using python:

- (a) Plot $R(E)$ and $T(E)$ for different values of V_0 .
(b) Mark the region $E < V_0 - m$.
(c) Plot the currents j_{in} , j_{ref} , j_{tr} and check their balance.

8. Discussion

- (a) Where exactly does the “paradox” arise if the interpretation is wrong?
(b) Why is the Klein paradox not a violation of causality?
(c) Which parts of this analysis require second quantization, and which do not?

9. References

- B. R. Holstein, *Klein’s paradox*, Am. J. Phys. **66**, 507 (1998).
- P. A. M. Dirac, *The Principles of Quantum Mechanics*.
- W. Greiner, *Relativistic Quantum Mechanics*.

6. The Dirac equation in 1+1 dimensions

Consider the Dirac equation in one space and one time dimension.

1. Find an explicit form of the γ^μ matrices satisfying

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}.$$

2. Find γ^5 .
3. Assuming that the fermion is massless, show that one obtains two independent equations of motion, for particles moving to the left and to the right.
4. Show that the currents are conserved:

$$\partial_\mu \left(\bar{\psi} \gamma^\mu \left(\frac{1 - \gamma^5}{2} \right) \psi \right) = 0, \quad \partial_\mu \left(\bar{\psi} \gamma^\mu \left(\frac{1 + \gamma^5}{2} \right) \psi \right) = 0.$$

5. Switch on an electric field A^1 that is constant in space and varies slowly in time. Find the Hamiltonian and its eigenvalues, assuming that the x axis has a finite length L and imposing periodic boundary conditions. Plot the energy spectrum for left- and right-handed fermions.
6. What happens to the energy spectrum if A^1 is slowly changed by

$$\Delta A^1 = \frac{2\pi}{eL}?$$

7. Relativistic tremor and pressure

1. Let the state of a free electron at $t = 0$ be given by

$$\psi(0, \mathbf{x}) = \frac{1}{(2\pi d^2)^{3/4}} e^{-\frac{\mathbf{x}^2}{4d^2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (1)$$

Find $\psi(t, \mathbf{x})$ for $t > 0$.

2. Obtain an analytic expression for the mean current as an integral over the 3-momentum:

$$\mathbf{J} = \int d^3x \psi^\dagger(x) \boldsymbol{\alpha} \psi(x).$$

3. Compute this integral and make an animation of $\mathbf{J}(t)$.
4. Does the tremor of the current last forever?
5. Compute the pressure P of the field and make an animation of $P(t)$.
6. Explain the results.

8. The Dirac γ matrices

1. Show that the commutator $[\sigma_{\mu\nu}, \sigma_{\alpha\beta}]$ can be written in terms of the matrices $\sigma_{\mu\nu}$. Find the coefficients of this expansion.

2. Compute the trace

$$\text{Tr} \left(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\alpha \gamma_\beta \gamma_5 \right).$$

3. Compute

$$\sigma_{\mu\nu} \sigma^{\mu\nu}, \quad \sigma_{\mu\nu} \gamma^5 \sigma^{\mu\nu}, \quad \sigma_{\mu\nu} \gamma^\alpha \sigma^{\mu\nu}.$$

9. The Stern–Gerlach experiment

Goal: Study the trajectories of a spin-1/2 particle in an inhomogeneous magnetic field $\mathbf{B} = \nabla\Phi(x, y, z)$ using the Pauli equation

$$i\frac{\partial\psi}{\partial t} = \left(\frac{(\hat{\mathbf{P}} - q\mathbf{A})^2}{2m} - \frac{q}{2m} \boldsymbol{\sigma} \cdot \mathbf{B} \right) \psi \quad (2)$$

and Ehrenfest's theorem, and analyze how the initial spin, the inhomogeneity of the field and the initial parameters of the particle affect its motion.

1. **Equations of motion:** Use Ehrenfest's theorem to compute the mean values of the coordinates and momentum:

$$\frac{d\langle \mathbf{r} \rangle}{dt} = \frac{\langle \mathbf{p} \rangle}{m}, \quad \frac{d\langle \mathbf{p} \rangle}{dt} = \langle \nabla(\boldsymbol{\mu} \cdot \mathbf{B}) \rangle,$$

where $\boldsymbol{\mu} = \frac{q}{2m} \boldsymbol{\sigma}$ is the magnetic moment of the particle (the Lorentz force is neglected here).

2. **Initial conditions:**

- (a) initial coordinates: $\langle \mathbf{r}(0) \rangle = (0, 0, 0)$;
- (b) initial momentum: $\langle \mathbf{p}(0) \rangle = (p, 0, 0)$;
- (c) initial spin: along the $+z$ axis ($\sigma_z = +1$) or the $-z$ axis ($\sigma_z = -1$).

3. **Magnetic field:** Consider an inhomogeneous magnetic field given, for example, by the scalar potential $\Phi(x, y, z)$:

$$\mathbf{B} = \nabla\Phi(x, y, z), \quad \Phi(x, y, z) = B_0 z + \frac{B_1}{2}(x^2 - y^2).$$

This field approximately models a real Stern–Gerlach apparatus.

4. **Trajectories:** Solve numerically the equations of motion for the mean coordinates $\langle x(t) \rangle$, $\langle y(t) \rangle$, $\langle z(t) \rangle$. Plot the trajectories of particles with spin along $+z$ and $-z$.

5. **Plots:**

- (a) the trajectories $\langle x(t) \rangle$ and $\langle z(t) \rangle$ for different values of the initial momentum p , the field strength B_0 and the gradient B_1 ;
- (b) a comparison of the deflections of particles with different initial spins ($+z$ and $-z$).

6. **Analysis:** Explain how the inhomogeneity of the field (B_1) and the parameters of the particle (p , m , initial spin) affect the trajectories. Explain why the Lorentz force does not affect the separation of the trajectories in this experiment.

10. The quantum eraser and delayed choice

Goal of the project: Work out mathematically and simulate numerically one of the most counterintuitive experiments in quantum mechanics. Show how which-path information about the particle destroys the interference and how correlations between events make it possible to “recover” it without violating causality.

The project is based on a modified double-slit experiment in which the coordinate x of the particle on the screen is entangled with its internal degree of freedom (for example, its spin).

1. Theory: entanglement and phase

Let the states of a particle that has passed through slit A or slit B be described by the functions $\psi_A(x)$ and $\psi_B(x)$. The state of the system in front of the screen is

$$|\Psi(x)\rangle = \frac{1}{\sqrt{2}} (\psi_A(x)|\uparrow\rangle + \psi_B(x)|\downarrow\rangle).$$

1. **Loss of interference** Compute the probability density of detecting the particle on the screen,

$$P(x) = \langle\Psi(x)|\Psi(x)\rangle.$$

Show that the interference term vanishes because the spin states are orthogonal ($\langle\uparrow|\downarrow\rangle = 0$). Explain the physical meaning: why does information that *in principle* can be extracted from the spin destroy the wave pattern?

2. **Change of basis (the quantum eraser)** Suppose that we measure the spin in the diagonal basis

$$|+\rangle = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}, \quad |-\rangle = \frac{|\uparrow\rangle - |\downarrow\rangle}{\sqrt{2}}.$$

Rewrite the state $|\Psi(x)\rangle$ in this basis and find the joint probability densities $P(x, \pm) = |\langle\pm|\Psi(x)\rangle|^2$ for the subsamples of particles.

3. **Recovering the pattern** Show that the distributions $P(x, +)$ and $P(x, -)$ contain interference terms, shifted in phase by π . Prove that

$$P(x, +) + P(x, -) = P(x).$$

2. Practical part: a Monte Carlo simulation

An experimentalist needs to understand how the “magic” turns into statistics.

1. **Generating the data** Using python, generate a sample of 10^5 events. For each event:
 - (a) draw the coordinate x from the distribution $P(x)$;
 - (b) for this x , draw the result of the spin measurement (+ or -) with probabilities $P(x, +)/P(x)$ and $P(x, -)/P(x)$.
2. **Sorting the events** Plot the histogram of all x . Then, using the results of the spin measurements, plot two separate histograms for the “plus events” and the “minus events”. Show how two interference patterns emerge from a featureless distribution.

3. Discussion: causality and “erasing the past”

1. **The delayed-choice paradox** Suppose that the spin is measured *after* the particle has hit the screen. Explain why this is not a signal sent from the future into the past.
2. **The no-signaling theorem** Discuss whether an observer at the screen can learn instantly which basis a colleague at the other end of the laboratory has chosen for the spin measurement.
3. **Interpretation** Why is the phrase “we erased the which-path information and the particle changed its mind” methodologically harmful for a research physicist?

4. What to submit

1. An analytic calculation of the densities $P(x, \pm)$.
2. A python script with plots: the total distribution (without interference) and the two correlated histograms (with interference).
3. A short essay (half a page) resolving the causality paradox.

References

- Y.-H. Kim, R. Yu, S.P. Kulik, Y. Shih, *A Delayed Choice Quantum Eraser*, Phys. Rev. Lett. **84**, 1 (2000).
- R. P. Feynman, R. B. Leighton, M. Sands, *The Feynman Lectures on Physics*, Vol. 3 (Quantum Mechanics).

11. Symmetries of the Dirac equation

1. Using Noether's theorem, obtain the conserved current for

$$\mathcal{L} = i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi$$

under the global gauge transformations

$$\psi \rightarrow e^{i\alpha}\psi, \quad \bar{\psi} \rightarrow e^{-i\alpha}\bar{\psi}.$$

2. Using Noether's theorem, obtain the conserved current for $\mathcal{L} = i\bar{\psi}\not{\partial}\psi$ under the chiral transformation

$$\psi \rightarrow e^{i\alpha\gamma^5}\psi.$$

Find how the field $\bar{\psi}$ transforms.

3. Prove that the action of the massless Dirac field is invariant under dilatations:

$$x \rightarrow x' = e^{-\rho}x, \quad \psi(x) \rightarrow \psi'(x') = e^{3\rho/2}\psi(x).$$

Find the corresponding Noether current and charge.

12. Symmetries of the σ model

The Lagrangian of the σ model:

$$\mathcal{L} = \frac{1}{2} [(\partial_\mu \sigma)(\partial^\mu \sigma) + (\partial_\mu \boldsymbol{\pi}) \cdot (\partial^\mu \boldsymbol{\pi})] + i\bar{\Psi}\not{\partial}\Psi + g\bar{\Psi}(\sigma + i\boldsymbol{\tau} \cdot \boldsymbol{\pi}\gamma_5)\Psi - \frac{m^2}{2}(\sigma^2 + \boldsymbol{\pi}^2) - \frac{\lambda}{4}(\sigma^2 + \boldsymbol{\pi}^2)^2,$$

where

- $\sigma(x)$ is a scalar field,
- $\boldsymbol{\pi}(x)$ is a three-component scalar field,
- $\Psi(x)$ is a doublet of fermion fields,
- $\boldsymbol{\tau}$ are the Pauli matrices.

Symmetry: Prove that the Lagrangian $\mathcal{L}$ has the following symmetry:

$$\sigma(x) \rightarrow \sigma(x),$$

$$\boldsymbol{\pi}(x) \rightarrow \boldsymbol{\pi}(x) - \boldsymbol{\alpha} \times \boldsymbol{\pi}(x),$$

$$\Psi(x) \rightarrow \Psi(x) + i\frac{\boldsymbol{\alpha} \cdot \boldsymbol{\tau}}{2}\Psi(x),$$

where $\boldsymbol{\alpha}$ is a 3-vector of infinitesimal parameters of the transformation. Find the corresponding Noether current.

13. Spectral flow and the chiral anomaly in 1+1

Goal: Understand the origin of the chiral anomaly as a geometric effect related to the spectral flow of energy levels.

1. Consider the massless Dirac equation in 1+1,

$$i\gamma^\mu(\partial_\mu + ieA_\mu)\psi = 0,$$

on a ring of length L .

2. Impose periodic boundary conditions and find the energy spectrum $E_n(A_1)$.

3. Change $A_1(t)$ slowly from 0 to

$$\Delta A_1 = \frac{2\pi}{eL}.$$

4. Plot the evolution of the energy levels and show that the levels cross zero.
5. Show that the numbers of right- and left-handed states change, even though the chiral current is formally conserved.
6. Discuss why this cannot be explained without taking the vacuum into account, even in a single-particle setting.

14. The Casimir effect

1. Consider the Casimir effect in 3D. Compute the pressure with which the plates attract each other, using the Euler–Maclaurin formula.
2. Explain the nature of the attraction between the plates. Can energy be extracted from the vacuum in this way?
3. Estimate the pressure. Compare it with the gravitational attraction of the plates as a function of the distance between them.
4. Review the experimental methods and results of measurements of the Casimir effect.

15. The dynamical Casimir effect

Goal: Study the creation of real quanta from the vacuum under nonstationary boundary conditions.

1. Consider a one-dimensional scalar field $\phi(t, x)$ with the boundary conditions

$$\phi(t, 0) = \phi(t, L(t)) = 0,$$

where

$$L(t) = L_0(1 + \epsilon \sin \omega t).$$

2. Find the instantaneous modes of the field and their frequencies $\omega_n(t)$.
3. Show that under the resonance condition

$$\omega \approx 2\omega_n$$

the number of quanta grows exponentially.

4. Compute numerically the radiated energy as a function of time.
5. Discuss the physics: where the energy of the created quanta comes from (the work of the external force that moves the boundary) and why energy is conserved.

16. Diffraction by an aperture

1. Consider the scattering of the scalar field $\phi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}$, where $\mathbf{k} = (0, 0, k)$, within the Helmholtz equation.
2. Find the Green's function of the Helmholtz equation.
3. Consider a small aperture in an opaque screen shaped as (a) a square and (b) an equilateral triangle.
4. Compute numerically the field behind the screen in the far zone.

17. The tachyon

A tachyon is a hypothetical particle that moves faster than light. This is often interpreted as a possibility to send superluminal signals. Investigate whether this is so.

1. Consider a chain of pendulums in a uniform gravitational field, suspended from a common support and connected by springs. The distance between neighboring pendulums is a .
2. Show that the Lagrangian of this system has the form

$$L = \sum_i \left(A \dot{\phi}_i^2 - B(\phi_i - \phi_{i-1})^2 - C(1 - \cos \phi_i) \right),$$

where ϕ_i is the angle by which the i th pendulum is deflected from equilibrium. Find A , B , C .

3. Pass to an infinite number of pendulums in the limit $a \rightarrow 0$. Show that the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 - m^2(1 - \cos \phi).$$

4. Obtain the equation of motion of the field ϕ :

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + m^2 \sin \phi = 0.$$

5. For small deviations from the stable equilibrium $\phi = 0$ one obtains the ordinary Klein–Fock–Gordon equation:

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + m^2 \phi = 0.$$

6. For small deviations from the unstable equilibrium $\phi = \pi$ one obtains the Klein–Fock–Gordon equation with a negative mass squared:

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - m^2 \phi = 0.$$

Formally, a plane wave $\phi = e^{-i\omega t + ikx}$ gives

$$\omega^2 = k^2 - m^2, \quad \text{and the group velocity } v_g = \frac{\partial \omega}{\partial k} = \frac{k}{\omega} > 1.$$

7. Show that the solution of the equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - m^2 \right) \phi = \delta(t) \delta(x)$$

propagates no faster than light: the retarded Green's function vanishes outside the light cone.

18. The Berry phase

A short theoretical summary: the geometric phase and its gauge structure

Consider a Hamiltonian $H(\mathbf{R})$ that depends on a set of external parameters $\mathbf{R}$, and its eigenstates

$$H(\mathbf{R}) |n(\mathbf{R})\rangle = E_n(\mathbf{R}) |n(\mathbf{R})\rangle,$$

assuming that the level E_n is nondegenerate.

If the parameters $\mathbf{R}(t)$ change slowly (adiabatically), then for the initial state $|n(\mathbf{R}(0))\rangle$ the solution of the Schrödinger equation has the form

$$|\psi(t)\rangle = e^{i\alpha_n(t)} |n(\mathbf{R}(t))\rangle,$$

where the phase $\alpha_n(t)$ satisfies

$$\dot{\alpha}_n(t) = -E_n(t) + i\langle n(\mathbf{R}(t)) | \dot{n}(\mathbf{R}(t)) \rangle.$$

Integrating over time, we obtain

$$\alpha_n(T) = - \int_0^T E_n(t) dt + \gamma_B,$$

where

$$\gamma_B = i \int_0^T \langle n(\mathbf{R}(t)) | \dot{n}(\mathbf{R}(t)) \rangle dt = i \oint_C \langle n | \nabla_{\mathbf{R}} n \rangle \cdot d\mathbf{R}.$$

The quantity

$$\mathcal{A}(\mathbf{R}) = i\langle n(\mathbf{R}) | \nabla_{\mathbf{R}} n(\mathbf{R}) \rangle$$

is called the *Berry gauge potential*. Under a phase transformation of the eigenstates

$$|n(\mathbf{R})\rangle \rightarrow e^{i\chi(\mathbf{R})} |n(\mathbf{R})\rangle$$

it transforms as

$$\mathcal{A} \rightarrow \mathcal{A} - \nabla_{\mathbf{R}} \chi,$$

exactly like the gauge potential of electrodynamics.

What is physically observable is not the potential itself but its curvature

$$\mathcal{F} = \nabla_{\mathbf{R}} \times \mathcal{A},$$

which is invariant under gauge transformations. For a closed contour C the Berry phase is the flux of this curvature:

$$\gamma_B = \int_S \mathcal{F} \cdot d\mathbf{S},$$

where S is a surface spanned by the contour C in parameter space.

Thus the geometric phase is determined solely by the geometry of the path in parameter space and does not depend on how fast the path is traversed.

Goal: Construct a measurable protocol for the Berry phase of a spin $\frac{1}{2}$ in a slowly varying magnetic field, show that the geometric phase is a consequence of the gauge structure of the space of eigenstates, separate it from the dynamical phase and check the conditions of adiabaticity numerically.

1. Two paths in parameter space: the same dynamics, different geometry

Consider a magnetic field of fixed magnitude B_0 with direction

$$\mathbf{n}(t) = \frac{\mathbf{B}(t)}{B_0}, \quad |\mathbf{n}(t)| = 1,$$

and the Hamiltonian

$$H(t) = -\gamma B_0 \mathbf{n}(t) \cdot \mathbf{S}.$$

Construct two cyclic evolutions $\mathbf{n}_A(t)$ and $\mathbf{n}_B(t)$ on the sphere of directions such that

- (a) the solid angles $\Omega_A \neq \Omega_B$ differ;
- (b) the dynamical phase is the same:

$$\int_0^T E_0(t) dt \text{ is the same for paths } A \text{ and } B.$$

Remark. One can take two cones with different θ and choose the times so that $\int_0^T |B(t)| dt$ coincide (or fix B_0 and take the same T).

2. An interference protocol

Consider a Ramsey-like protocol for a two-level system (a spin):

- (a) prepare a superposition of two eigenstates of $H(0)$;
- (b) realize path A in one branch (arm of the interferometer) and path B in the other;
- (c) at the end, project onto the initial basis and measure the interference signal.

Formulate how the observed interference phase depends on

$$\Delta\gamma = \gamma_B(A) - \gamma_B(B).$$

Explain why it is *the difference* of geometric phases that is physically observable, while the “absolute” phase depends on the choice of gauge of the eigenvectors.

3. Gauge structure: the Berry connection and a “magnetic monopole” in parameter space

Define the Berry gauge potential of the ground state:

$$\mathcal{A}(\mathbf{n}) = i\langle 0(\mathbf{n}) | \nabla_{\mathbf{n}} | 0(\mathbf{n}) \rangle.$$

- (a) Show that under a change of the phase convention $|0\rangle \rightarrow e^{i\chi(\mathbf{n})}|0\rangle$ the potential changes as $\mathcal{A} \rightarrow \mathcal{A} - \nabla_{\mathbf{n}}\chi$.
- (b) Compute the curvature (the Berry field)

$$\mathcal{F} = \nabla_{\mathbf{n}} \times \mathcal{A},$$

and interpret it as the field of a “monopole” in the space of directions $\mathbf{n}$.

- (c) Choose two different gauges of the eigenstates (for example, singular on the northern and on the southern hemisphere of the sphere of directions $\mathbf{n}$) and show that
 - i. the Berry potential $\mathcal{A}$ depends on the choice of gauge;
 - ii. the curvature $\mathcal{F}$ and the phase γ_B for a closed contour do not change.

4. Numerical check: adiabaticity and transition errors

Solve numerically the Schrödinger equation

$$i \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle$$

for paths A and B .

- (a) Plot the probability to remain in the ground state,

$$P_0(t) = |\langle 0(t) | \psi(t) \rangle|^2.$$

- (b) Introduce the dimensionless adiabaticity parameter

$$\epsilon(t) = \frac{|\langle 1(t) | \dot{H}(t) | 0(t) \rangle|}{(E_1(t) - E_0(t))^2},$$

and check the relation: the smaller $\max_t \epsilon(t)$, the closer $P_0(T)$ is to 1.

- (c) For a fixed path (for example, a cone) study how the error

$$1 - P_0(T)$$

depends on the period T (logarithmic plot).

5. Extracting the geometric phase from the numerical solution

Extract numerically the total phase of the overlap

$$\Phi_{\text{tot}} = \arg \langle \psi(0) | \psi(T) \rangle,$$

and separate the dynamical part

$$\Phi_{\text{dyn}} = - \int_0^T \langle \psi(t) | H(t) | \psi(t) \rangle dt.$$

Define the numerical estimate of the geometric phase as

$$\Phi_{\text{geom}} = \Phi_{\text{tot}} - \Phi_{\text{dyn}},$$

and compare $\Phi_{\text{geom}}(A) - \Phi_{\text{geom}}(B)$ with the prediction in terms of the solid angles $\Omega_{A,B}$.

6. Observability and robustness

- (a) Add a small noise to the field direction, $\mathbf{n}(t) \rightarrow \mathbf{n}(t) + \delta \mathbf{n}(t)$, followed by normalization to the unit sphere. Check that the result depends mainly on the geometry of the contour (the solid angle) and not on the details of its parametrization in time.
- (b) Discuss in what sense the geometric phase is “robust” and where this robustness ends (violation of adiabaticity, level crossings).

7. What to submit

- (a) A short theoretical derivation (1–2 pages): the interference protocol, the gauge, $\Delta\gamma$ as an observable.
- (b) Numerical plots: $P_0(t)$, $\epsilon(t)$, the dependence of $1 - P_0(T)$, and a comparison of Φ_{geom} with the geometric prediction.
- (c) A minimal git repository: code (python), a short README, the final $\LaTeX$ report.

8. References

- M. V. Berry, *Quantal phase factors accompanying adiabatic changes*, Proc. R. Soc. A **392**, 45 (1984).

19. Landau levels: from nonrelativistic particles to graphene

Goal of the project: Study the quantization of the motion of a charged particle in a uniform magnetic field. Show how the magnetic field leads to a discrete spectrum (Landau levels), how the choice of gauge affects the form of the wave functions, and compare the spectra of ordinary electrons and of massless Dirac fermions (the effective description of graphene).

The project connects standard quantum mechanics, condensed matter physics and the relativistic equations that underlie quantum field theory.

We consider the motion of a particle in the (x, y) plane in a uniform magnetic field $\mathbf{B} = B \hat{z}$.

1. Theory: the oscillator structure

In a magnetic field translation invariance is broken, and the free motion of the particle is quantized.

It is convenient to use the *Landau gauge*

$$\mathbf{A} = (0, Bx, 0).$$

1. The Hamiltonian and integrals of motion

Write the Hamiltonian of a charged particle:

$$H = \frac{1}{2m} [p_x^2 + (p_y - qBx)^2].$$

- (a) Show that the operator p_y commutes with the Hamiltonian.
- (b) Using the substitution $\psi(x, y) = e^{ik_y y} \phi(x)$, reduce the problem to a one-dimensional harmonic oscillator.

2. The spectrum of Landau levels and their degeneracy

- (a) Find the energy spectrum

$$E_n = \hbar \omega_c \left(n + \frac{1}{2} \right), \quad \omega_c = \frac{|q|B}{m}.$$

- (b) Find the position x_0 of the center of the oscillator.
- (c) Explain the physical meaning of the infinite degeneracy of the levels and its relation to the translation symmetry along the y axis.

3. The relativistic case: massless Dirac fermions (graphene)

Consider the Dirac equation for massless particles in a magnetic field, which effectively describes electrons in graphene:

$$H = v_F \boldsymbol{\sigma} \cdot (\mathbf{p} - q\mathbf{A}).$$

- (a) Show that the spectrum of the Landau levels is

$$E_n = \pm v_F \sqrt{2|q|B\hbar n}, \quad n = 0, 1, 2, \dots$$

- (b) Explain the qualitative difference between this spectrum and the nonrelativistic one.

2. Practical part: visualization and the density of states

The aim is to see the Landau levels and their physical consequences.

1. Wave functions

Plot the probability density $|\psi(x, y)|^2$ for the first few Landau levels.

- (a) Show how the position of the maximum depends on the wave vector k_y .
- (b) Illustrate the localization of states in a magnetic field.

2. Comparing the spectra

Plot the energies $E_n(B)$ for

- (a) a nonrelativistic electron;
- (b) a massless Dirac fermion (graphene).

Emphasize the difference between the linear and the square-root dependence on B .

3. Density of states (DOS)

Model the density of states by replacing the δ peaks of the Landau levels with narrow Gaussians (level broadening).

- (a) Plot the DOS for different values of B .
- (b) Show how the discrete structure disappears as $B \rightarrow 0$.

3. Discussion and the connection with quantum field theory

1. The quantum Hall effect

Explain how the gap between the Landau levels leads to the plateaus of the Hall conductance. What role does the degeneracy of the levels play?

2. The magnetic length

Introduce the magnetic length

$$\ell_B = \sqrt{\frac{\hbar}{|q|B}}.$$

Discuss its physical meaning and its role as the fundamental scale in problems with strong magnetic fields (graphene, neutron stars, QFT in strong fields).

3. The zero level and topology

In graphene there is a level with energy $E = 0$. Discuss its relation to chiral symmetry, spectral asymmetry and the topological properties of the system.

4. What to submit

- 1. A derivation of the Landau levels for the Schrödinger equation and for the Dirac equation.
- 2. Plots of the wave functions and of the spectra.
- 3. A numerical model of the density of states.

References

- L. D. Landau, E. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory* (§112).
- A. H. Castro Neto et al., *The electronic properties of graphene*, Rev. Mod. Phys. **81**, 109 (2009).
- D. Tong, *Lectures on the Quantum Hall Effect*, arXiv:1606.06687.

20. The Aharonov–Bohm effect and gauge invariance

Goal of the project: Understand why in quantum mechanics the *gauge potential* $\mathbf{A}$ is physically significant even though the classical force depends only on the magnetic field $\mathbf{B}$. Show that the Aharonov–Bohm effect is *topological* and is observed as a shift of the interference pattern.

We consider the interference of electrons in the presence of an infinitely long thin solenoid with magnetic flux Φ , while

$$\mathbf{B} = 0 \quad \text{in the region where the electrons move.}$$

1. Theory: the phase and the gauge potential

In classical electrodynamics a particle feels only the local field $\mathbf{B} = \nabla \times \mathbf{A}$. In quantum mechanics the phase structure of the wave function makes the vector potential $\mathbf{A}$ observable.

1. The Schrödinger equation with a vector potential

Consider the Schrödinger equation for a particle of charge q :

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m} (-i\hbar \nabla - q\mathbf{A})^2 \psi.$$

Explain why the potential $\mathbf{A}$ can affect the phase of the wave function even when $\mathbf{B} = 0$.

2. The phase factor

Let $\psi_0(\mathbf{r}, t)$ be a solution of the Schrödinger equation with $\mathbf{A} = 0$. Show that for nonzero $\mathbf{A}$ the solution can be written as

$$\psi(\mathbf{r}, t) = \exp\left(\frac{iq}{\hbar} \int_C \mathbf{A} \cdot d\mathbf{l}\right) \psi_0(\mathbf{r}, t),$$

where the integral is taken along the path of the particle.

3. The phase difference in an interferometer

Consider a double-slit experiment. Let the electron travel along two paths C_1 and C_2 that enclose the solenoid.

Show that the phase difference is

$$\Delta\varphi = \frac{q}{\hbar} \oint \mathbf{A} \cdot d\mathbf{l} = \frac{q\Phi}{\hbar}.$$

Explain why the result depends only on the magnetic flux Φ and not on the shape of the paths.

2. Practical part: simulating the interference

The aim is to show numerically that a change of the magnetic flux shifts the interference fringes.

1. The interference pattern

Model the intensity on the screen,

$$I(x) = I_0 \left[1 + \cos\left(\frac{2\pi d}{\lambda L} x + \frac{q\Phi}{\hbar}\right) \right],$$

where d is the distance between the slits, L is the distance to the screen and λ is the de Broglie wavelength of the electrons.

Plot $I(x)$ for several values of Φ .

Derive this formula.

2. Animation

Make an animation showing how the interference pattern shifts as the flux changes,

$$\Phi \in [0, \Phi_0], \quad \Phi_0 = \frac{h}{q}.$$

3. Monte Carlo

Simulate the arrival of individual electrons on the screen:

- (a) draw the coordinates of the hits from the distribution $I(x)$;
- (b) plot histograms for different values of Φ ;
- (c) show how the interference pattern builds up statistically.

3. Discussion and physical meaning

1. Gauge invariance

Explain why the phase of the wave function depends on the choice of gauge, while the observable phase difference $\Delta\varphi$ does not.

2. Topological character

Explain why the shift of the pattern is periodic in the flux with period Φ_0 (an integer number of flux quanta is unobservable; only the fractional part of Φ/Φ_0 matters) and is insensitive to local deformations of the paths.

4. What to submit

1. The theoretical derivation.
2. An animation of the interference pattern.
3. A minimal git repository with the code (python) and the report.

References

- Y. Aharonov, D. Bohm, *Significance of electromagnetic potentials in the quantum theory*, Phys. Rev. **115**, 485 (1959).